MS4414 Theoretical Mechanics
Tutorial week 9: Energy and Gravitation

Thursday 22 March 2011

Conservation of energy

- If there is no friction, energy is conserved

$$E = E^{kinetic} + E^{potential}$$

- Kinetic energy

$$E = \frac{1}{2}mv^2$$

- Potential energy

$$F = -\nabla E^{potential}$$

- Gravity

$$E^{potential} = -\frac{\mathcal{G}mM}{r}$$

Question 1a

- Forces

$$F_{12} = -\frac{\mathcal{G}m_1m_2}{(x_2-x_1)^2}\mathbf{e_{12}} \ , \quad F_{21} = -\frac{\mathcal{G}m_1m_2}{(x_2-x_1)^2}\mathbf{e_{21}} \ ,$$
$$F_{13} = -\frac{\mathcal{G}m_1m_3}{(x_3-x_1)^2}\mathbf{e_{13}} \ , \quad F_{31} = -\frac{\mathcal{G}m_1m_3}{(x_3-x_1)^2}\mathbf{e_{31}} \ ,$$
$$F_{23} = -\frac{\mathcal{G}m_2m_3}{(x_3-x_2)^2}\mathbf{e_{23}} \ , \quad F_{32} = -\frac{\mathcal{G}m_2m_3}{(x_3-x_2)^2}\mathbf{e_{32}} \ .$$

where $\mathbf{e_{12}}$ is the unit vector from m_1 to m_2.

Question 1b

- Potential energy

$$F_1 = F_{21} + F_{31} = -\frac{\partial U}{\partial x_1}\mathbf{i} \ ,$$
$$F_2 = F_{32} + F_{12} = -\frac{\partial U}{\partial x_2}\mathbf{i} \ ,$$
$$F_3 = F_{13} + F_{23} = -\frac{\partial U}{\partial x_3}\mathbf{i} \ .$$

- Careful with the **direction** of the unit vector.

Question 2

- Object at the distance
  - x_1 from planet 1
  - x_2 from planet 2

$$L = x_1 + x_2$$

- Equal forces

$$-\frac{\mathcal{G}m\ m_1}{x_1^2} = -\frac{\mathcal{G}m\ m_2}{x_2^2}$$

- Solution

$$x_1 = \frac{\sqrt{m_1}}{\sqrt{m_1} + \sqrt{m_2}} L \ , \quad x_2 = \frac{\sqrt{m_2}}{\sqrt{m_1} + \sqrt{m_2}} L \ .$$

- Kinetic Energy

$$E = \frac{1}{2} m v^2$$

- Calculate elementary energy

$$dE = \frac{1}{2} dm v^2 = \frac{\lambda}{2} dr \left(r\omega\right)^2$$

where $\lambda = m/L$ is the lineic mass.

- Total kinetic energy

$$E = \int_0^L dE = \frac{\lambda\omega^2}{2} \int_0^L r^2 dr = \frac{\lambda\omega^2}{2} \frac{L^3}{3} = \frac{mL^2\omega^2}{6}$$

Question 4

- Energy

$$E = E^{Kinetic} + E^{Potential} = \frac{1}{2}mv^2 - \frac{\mathcal{G}mM}{d}$$

- Conservation of energy

$$0 + 0 - 0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{\mathcal{G}m_1 m_2}{R_1 + R_2}$$

- Conservation of momentum

$$0 = m_1 \mathbf{v_1} + m_2 \mathbf{v_2} \Longrightarrow m_1 v_1 = m_2 v_2$$

- Solution

$$v_1 = m_2 \sqrt{\frac{2\mathcal{G}}{(m_1 + m_2)(R_1 + R_2)}}, \quad v_2 = m_1 \sqrt{\frac{2\mathcal{G}}{(m_1 + m_2)(R_1 + R_2)}}$$

Question 5

- Conservation of energy

$$E^{before} = E^{after}$$

- Kinetic energy

$$\frac{1}{2}mV^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \Longrightarrow V^2 = v_1^2 + v_2^2$$

- Conservation of momentum

$$\mathbf{V} = \mathbf{v_1} + \mathbf{v_2}$$

- Two options
  - Reciprocal of Pythagorean theorem
  - Vector analysis

$$\begin{aligned}\mathbf{V} = \mathbf{v_1} + \mathbf{v_2} \quad &\Longrightarrow \quad \mathbf{V}^2 = (\mathbf{v_1} + \mathbf{v_2})^2 \\ &\Longrightarrow \quad \mathbf{V}^2 = \mathbf{v_1}^2 + \mathbf{v_2}^2 + 2\mathbf{v_1}\cdot\mathbf{v_2} \\ &\Longrightarrow \quad \mathbf{v_1}\cdot\mathbf{v_2} = 0\end{aligned}$$

Question 6 Part 1

- Expression for ω

$$\omega = \frac{2\pi}{T}$$

- Position

$$(x(t), y(t)) = (d\cos(\omega t), d\sin(\omega t))$$

- Acceleration

$$(a_x, a_y) = \left(-d\omega^2\cos(\omega t), -d\omega^2\sin(\omega t)\right)$$

- Force

$$F = \left(-\frac{\mathcal{G}mM}{d^2}\cos(\omega t), -\frac{\mathcal{G}mM}{d^2}\sin(\omega t)\right)$$

Question 6 Part 2

- Newton's second law

$$F = ma$$

- Substitute

$$-dm\omega^2 \cos(\omega t) = -\frac{\mathcal{G}mM}{d^2}\cos(\omega t)$$
$$-dm\omega^2 \sin(\omega t) = -\frac{\mathcal{G}mM}{d^2}\sin(\omega t)$$

- Relationship

$$\omega = \sqrt{\frac{\mathcal{G}M}{d^3}} \Longrightarrow T = 2\pi\sqrt{\frac{d^3}{\mathcal{G}M}} = 58.8 \times 10^6 s$$

- Answer 682 days (real orbit=687 days according to Wikipedia)

Question 7

- Energy

$$E = E^{Kinetic} + E^{Potential} = \frac{1}{2}mv^2 - \frac{\mathcal{G}mM}{d}$$

- Conservation of energy

$$\frac{1}{2}mv_0^2 - 0 = \frac{1}{2}mv_1^2 - \frac{\mathcal{G}mM}{R}$$

- Relationship

$$v_1 = \sqrt{v_0^2 + \frac{2\mathcal{G}M}{R}}$$