Conservation of Energy

MS4414 Theoretical Mechanics

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1 Introduction

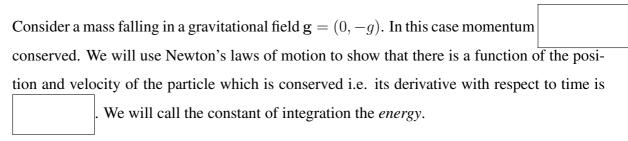
Conserved quantities are useful in mechanics: they allow us to make short-cuts in calculations.

For example we have already encountered conservation of momentum. The principle of conservation of momentum states that the momentum of a collection of particles is a conserved quantity, provided the particles do not interact with any force fields. In collisions the momentum of particles before and after the collisions are the statement of the particles of the particles after the collision without knowing any of the details of the forces acting between them (which are unbelievably complicated and still the subject of research).

Consevation of energy is similarly useful. Here we will look at conservation of energy in a uniform gravitational field. We will see that, if we are interested only in positions and velocities, conservation of energy offers a simpler way to solve problems.

We will revisit conservation of energy when we discuss elastic collisions, nonuniform gravitational fields, and the Lagrangian and Hamiltonian mechanics.

2 Uniform Gravitational field



Newton's second law states that

$$m\mathbf{g} = m\mathbf{a} \tag{1}$$

Take the dot product of both sides with the vector v, the velocity of the particle.

$$m\mathbf{v}\cdot\mathbf{g} = \boxed{ (2)}$$

Write ${\bf v}$ and ${\bf a}$ in terms of the displacement ${\bf s}$ of the particle.

$$m\mathbf{g} \cdot \left(\frac{\mathbf{d}\mathbf{s}}{\mathbf{d}t}\right) = m$$
 (3)

We can rewrite both sides as total derivatives and integrate (m and g are constants)

$$\frac{d}{dt}(m\mathbf{g} \cdot \mathbf{s}) = \frac{d}{dt} \left[\frac{1}{2} m \left(\frac{d\mathbf{s}}{dt} \right)^2 \right]$$
 (4)

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m\mathbf{g} \cdot \mathbf{s} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m\mathbf{v}^2 \right) \tag{5}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m \mathbf{v}^2 - m \mathbf{g} \cdot \mathbf{s} \right) \tag{6}$$

where $\mathbf{v}^2 =$

As promised, we have a quantity whose derivative is zero. This means the quantity is

Integrate to obtain a constant of integration I shall call the energy, E.

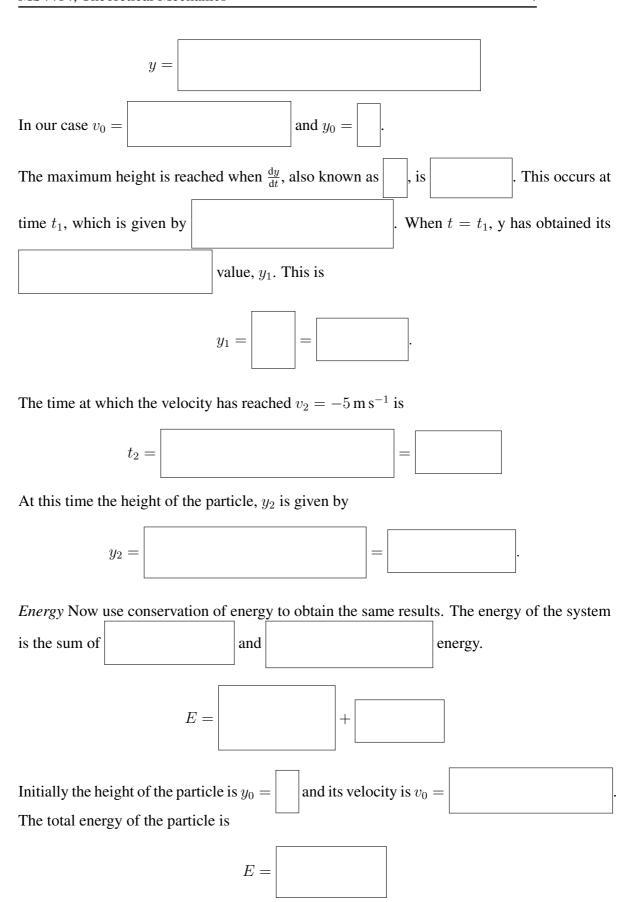
$$E = \frac{1}{2}m\mathbf{v}^2 - m\mathbf{g} \cdot \mathbf{s} \tag{7}$$

Writing this out in component form with $\mathbf{g}=(0,-g),\,\mathbf{v}=(v_x,v_y),\,\mathbf{s}=(x,y)$

$$E = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + mgy\tag{8}$$

 $\frac{1}{2}m\mathbf{v}^2$ is called **kinetic energy**, and mgy is called **potential energy**.

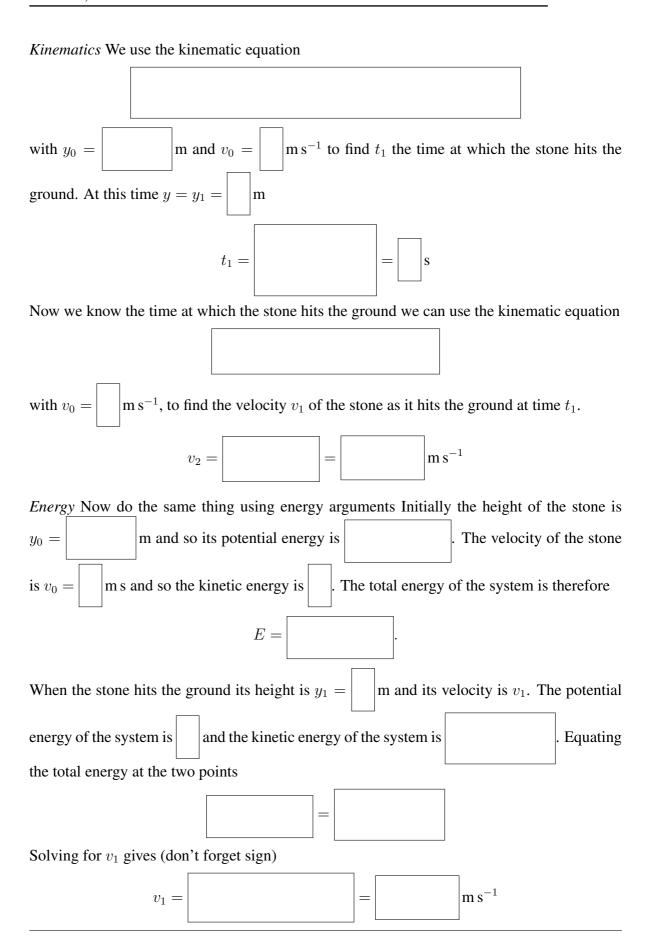
Worked Example On Earl 5 m. Show that these involve	•	$0\mathrm{ms^{-1}}$ and pole vault up to about
The kinetic energy of a spring		elocity v is $\overline{}$. Tak-
The inneric energy of a sprint	or mass meranning wan to	. Tun
ing a typical mass of a sprint	er as	, the kinetic energy gener-
ated by the sprinter is		•
The potential energy of a pole-vaulter at the top of the jump is Again, taking		
the mass of the athlete as		and acceleration due to gravity as
	, the potential energ	gy of the athlete is
Therefore the kinetic energy	of the sprinter and the poter	ntial energy of the pole-vaulter are
Worked Example A stone is thrown up in the air with initial velocity $v=10\mathrm{ms^{-1}}$. What is the maximum height reached by the stone? How high is the stone when its velocity is $-5\mathrm{ms^{-1}}$? Take $g=10\mathrm{ms^{-1}}$ and neglect air resistance. First we will solve this problem the old fashioned way using the kinematic equations. Then we will check that using conservation of energy gives the same answers.		
_		v of an object thrown directly up-
wards in a	gravitational field g are	2
	v =	



When the particle has reached its maximum height, y_1 the velocity of the particle is $v_1 =$ Therefore the kinetic energy of the particle is The potential energy must be equal to the Therefore $E = \frac{1}{2}mv_0^2 =$ Solving for y_1 gives m $y_1 =$ At height y_2 the velocity of the particle is $v_2 = -5 \,\mathrm{m\,s^{-1}}$. At that point the kinetic energy of the particle is and the potential energy is The sum of the potential and kinetic energies is the therefore Solving this equation for y_2 gives m $y_2 =$

Worked Example A stone is dropped from from a height of 125 m. What is its velocity when it hits the ground? Take $g = 10 \, \text{m s}^{-1}$ and neglect air resistance.

Again, first solve this using the kinematic equations and then use energy arguments.



Kinematics or Conservation of Energy? Where possible use conservation of energy: it is simpler. In the examples above we only used one energy equation, but needed two kinematic equations. However, it is not possible to use energy arguments to work out times—the whole point if the conservation of energy is time independence. So if the question requires you to work out, or use, a time you will have to use the kinematic equations.

In an exam (or if large sums of money or someones life depend on you getting the correct answer) calculate both ways as a check on the answer—if you have time.