Partial Fractions

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1 Top Heavy

To expand into partial fractions, the order of the polynomial in the numerator must be one order lower that that of the denominator. Use polynomial long devision to reduce the order.

$$\frac{x^3 + x^2 + x + 1}{x^2 - 3x + 2} = x + 4 + \frac{11x - 7}{x^2 - 3x + 2} \tag{1}$$

Now we can convert the remainder to partial fractions

$$x + 4 + \frac{11x - 7}{x^2 - 3x + 2} = x + 4 + \frac{15}{x - 2} - \frac{4}{x - 1} \tag{2}$$

In this form the expression can easily be integrated

$$\int \frac{x^3 + x^2 + x + 1}{x^2 - 3x + 2}\,\mathrm{d}x = \frac{x^2}{2} + 4x + 15\ln(x - 2) - 4\ln(x - 1) \tag{3}$$

2 Standard

The standard form is easy to factorise

$$\frac{5x + 2}{x^2 - x - 2} = \frac{1}{x + 1} + \frac{4}{x - 2} \tag{4}$$

And simple to integrate

$$\int \frac{5x + 2}{x^2 - x - 2}\,\mathrm{d}x = \ln(x + 1) + 4\ln(x - 2) \tag{5}$$

3 Repeated Root

The polynomial in the denominator may contain a repeated root

$$\frac{x^2+x+1}{x^3-3x^2+4} = \frac{1}{9\left(x+1\right)} + \frac{8}{9\left(x-2\right)} + \frac{7}{3\left(x-2\right)^2} \tag{6}$$

Integrating

$$\int \frac{x^2+x+1}{x^3-3x^2+4}\,\mathrm{d}x = \tfrac{1}{9}\ln\left(x+1\right) + \tfrac{8}{9}\ln\left(x-2\right) - \frac{7}{3\left(x-2\right)} \tag{7}$$

4 Unfactorable Part

Sometimes the denominator cannot be completely factorised.

$$\frac{x^2+4x+1}{x^3+2x^2+2x+1} = -\frac{2}{x+1} + \frac{3x+3}{x^2+x+1} \tag{8}$$

Integration is still possible

$$\begin{aligned}\int \frac{x^2+4x+1}{x^3+2x^2+2x+1}\,\mathrm{d}x \\ = -2\ln\left(x+1\right) + \tfrac{3}{2}\ln\left(x^2+x+1\right) + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\end{aligned} \tag{9}$$