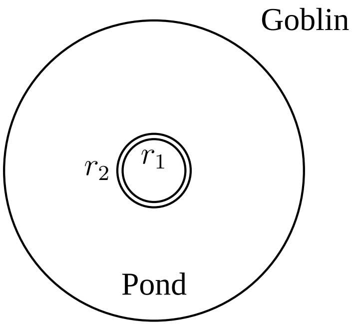
Circular Motion

MS4414 Theoretical Mechanics

William Lee

Contents

Microsoft Hiring Question You are in a boat in the exact centre of a perfectly circular lake. There is a goblin on the shore of the lake. The goblin wants to do bad things to you. The goblin can't swim and doesn't have a boat. Provided you can make it to the shore—and the goblin isn't there, waiting to grab you—you can always outrun him on land and get away. The problem is this: The goblin can run four times as fast as the maximum speed of your boat. He has perfect eyesight, never sleeps and is extremely logical. He will do everything in his power to catch you. How would you escape the goblin?



If the radius of the pond is r. If the goblin is on top of the opposite side of the pond to the boat the boat can escape if it is a distance r_1 or greater. $4(r - r_1) = \pi r \implies r_1 = \frac{4-\pi}{4} r \approx 0.22r$. The boat can circle faster than the goblin within a circle of r_2. $4r_2 = r \implies r_2 = \frac{1}{4} r$

Therefore you should circle the boat between r_1 and r_2 until goblin and boat are diametrically opposed. At that point it is safe to head towards the edge of the point.

1 Uniform Circular Motion

A particle undergoing uniform circular motion has Cartesian coordinates

$$x = R\cos(\omega t + \theta_0) \qquad y = R\sin(\omega t + \theta_0),$$

and polar coordinates

$$r = R \qquad \theta = \omega t + \theta_0$$

Angular Velocity Angular velocity, usually denoted ω is the rate of change of the angular coordinate, θ.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$$

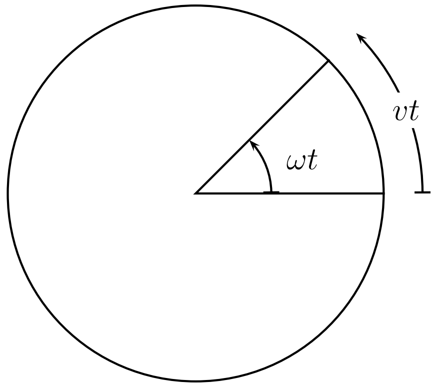
Angular Frequency Angular frequency is the inverse of the time for a single revolution.

Speed the magnitude of the particle's (linear) velocity.

If it takes a particle T seconds to make a complete revolution round a circle of radius R:

- The *angular frequency* is T^{-1}
- In T the particle travels a *distance* of $2\pi R$, therefore its speed is $2\pi R/T$

- There are 2π radians in a complete revolution so the particle moves through an angle of 1 radian in $T/2\pi$. Its angular velocity is $\omega = 1/\left(T/2\pi\right) = 2\pi/T$
- Angular velocity, ω, and speed, v, are related by $v = r\omega$.



Worked example *Which is greater, the velocity of the the Earth's rotation (at the equator) or the velocity of the Earth's orbit?*

The Earth, which has radius $r_{\mathrm{E}} = 6.4 \times 10^6$ m, completes one rotation in 23 hours 56 minutes and 4 seconds i.e. $T_{\mathrm{day}} = 8.62 \times 10^4$ s.

$$
\begin{aligned}
\text{angular frequency} &= \frac{1}{T_{\text{day}}} = \frac{1}{8.62 \times 10^4} = 1.16 \times 10^{-5}\,\text{Hz} \\
\text{angular velocity} &= \frac{2\pi}{T_{\text{day}}} = \frac{2\pi}{8.62 \times 10^4} = 7.29 \times 10^{-5}\,\text{rad}\,\text{s}^{-1} \\
\text{linear velocity} &= \text{radius} \times \text{angular velocity} \\
&= \left(6.4 \times 10^6\right) \times \left(7.29 \times 10^{-5}\right) = 467\,\text{m}\,\text{s}^{-1}
\end{aligned}
$$

The Earth is $R_{\mathrm{ES}} = 1\,\mathrm{AU} = 1.5 \times 10^{11}$ m away from the Sun (on average). It completes an orbit in 1 year i.e. $T_{\mathrm{yr}} = 3.16 \times 10^7$ s,

$$
\begin{aligned}
\text{angular frequency} &= \frac{1}{T_{\text{yr}}} = \frac{1}{3.16 \times 10^7} = 3.17 \times 10^{-8}\,\text{Hz} \\
\text{angular velocity} &= \frac{2\pi}{T_{\text{yr}}} = \frac{2\pi}{3.16 \times 10^7} = 1.99 \times 10^{-8}\,\text{rad}\,\text{s}^{-1} \\
\text{linear velocity} &= \text{radius} \times \text{angular velocity} \\
&= \left(1.5 \times 10^{11}\right) \times \left(1.99 \times 10^{-8}\right) = 2.98 \times 10^4\,\text{m}\,\text{s}^{-1}
\end{aligned}
$$

Conclusion: The velocity due to the Earth's orbit is greater.

2 Circular motion and Vectors

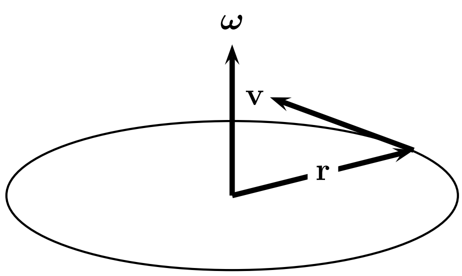
A small angle $\delta\theta$ may be represented by an vector of magnitude $\delta\theta$ and direction normal to the plane of rotation: $\delta\boldsymbol{\theta}$. (A large angle cannot be represented by a vector. Why not?)

Since

$$\omega = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

only involves a small angle $\delta\theta = \theta(t + \delta t) - \theta(t)$, we can convert the above into a vector equation.

$$\boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}.$$



We can now write a vector equation for the velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

Remember that $\mathbf{r}$ will be changing with time, so the equation could be written as

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{r}$$

3 Circular motion and complex numbers

Circular motion on a plane can be modelled by taking x and y to be the real and imaginary parts of the complex number

$$z = Re^{i(\omega t + \theta_0)}.$$

It is easy to see that this corresponds to the Cartesian formula for circular motion

$$x = \Re z = R\cos(\omega t + \theta_0)$$
$$y = \Im z = R\sin(\omega t + \theta_0)$$

3.1 Modelling the solar system

On interesting case where objects move in (more or less) circular orbits on a plane is the orbits of the planets of the solar system (now Pluto has been demoted).

The clever part is that move the origin to another planet (e.g. Earth) and see the motions of the other planets relative to that planet all I have to do is subtract the complex number describing the Earth's orbit from the complex number describing the planet's orbit.

The Dresden Codex The Dresden codex was left by the Mayan people of central America. The key to deciphering it was the number 584.

4 Uniform Circular Motion and Acceleration

If we take another look at the vector formula relating velocity and angular velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

we see that we can differentiate it to get an acceleration

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{\omega}) \times \mathbf{r}}{\mathrm{d}t}$$

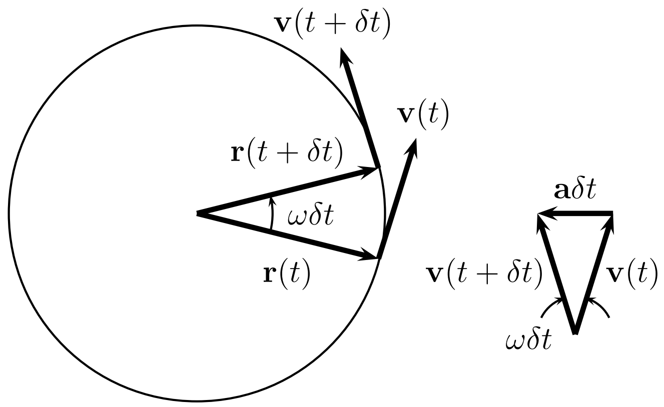
For uniform circular motion $\boldsymbol{\omega}$ is a constant, but $\mathbf{r}$ is constantly changing $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$.

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v}$$

The acceleration is orthogonal to both the velocity and the angular velocity: therefore it must be parallel to the radius (but directed inwards). The magnitude of the acceleration can be written as

$$a = \omega v = \frac{v^2}{r} = \omega^2 r$$

We can see this geometrically.



$$\mathbf{a}\delta t = \mathbf{v}(t+\delta t) - \mathbf{v}(t)$$

where $\mathbf{a}$ is directed in the inwards radial direction. The magnitude of the acceleration is

$$a = v\omega$$

Alternatively and rather neatly, we can calculate the acceleration from the complex number formula. If the components of $\mathbf{r}$ are the real and imaginary parts of the complex number $z = Re^{i\omega t}$, then the velocity is given by

$$\dot{z} = i\omega Re^{i\omega t} = i\omega z$$

The factor of i tells us the velocity, $\dot{z}$, is at 90° to the radius, z. The magnitude of the velocity is given by $|\dot{z}| = R\omega$.

The acceleration is given by

$$\ddot{z} = -\omega^2 Re^{i\omega t}$$

i.e. the magnitude of the acceleration is $|\ddot{z}| = R\omega^2$ and it is directed in the opposite direction to the radius, z.

We will return to acceleration of uniform circular motion when we have learned about Newton's laws of motion and gravitation.

5 Angular Acceleration

If the angular frequency $\boldsymbol{\omega}$ is changing at a constant rate $\boldsymbol{\alpha}$ then:

$$\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boxed{\boldsymbol{\alpha}}$$

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} = \boldsymbol{\omega} = \boxed{\boldsymbol{\alpha}t + \boldsymbol{\omega}_0}$$

$$\boldsymbol{\theta} = \boxed{\tfrac{1}{2}\boldsymbol{\alpha}t^2 + \boldsymbol{\omega}_0 t + \boldsymbol{\theta}_0}$$

where, at $t = 0$, the angle is $\boldsymbol{\theta}_0$, and the angular velocity is $\boldsymbol{\omega}_0$.

Worked Example Tidal friction in the Earth-Moon system transfers angular momentum from the Earth to the Moon: the length of the day on Earth is increasing by 2.4 milliseconds per century and the Moon is moving away from the Earth at a rate of 38.14 millimetres per year. What is the angular acceleration of the Earth?

Calculate two ways

$$\begin{aligned}\alpha &= \frac{\mathrm{d}\omega}{\mathrm{d}t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi}{T}\right)\\ &= -\frac{2\pi}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t}\end{aligned}$$

where T is the time of one day $T = 8.64 \times 10^4$ s and $\frac{\mathrm{d}T}{\mathrm{d}t} = 7.61 \times 10^{-13}$

$$\alpha = -\frac{2\pi}{(8.64 \times 10^4)^2} \times 7.61 \times 10^{-13} = -6.4 \times 10^{-22}\,\mathrm{rad\,s^{-2}}$$

Alternatively

$$\begin{aligned}T &= \frac{2\pi}{\omega}\\ \frac{\mathrm{d}T}{\mathrm{d}t} &= -\frac{2\pi}{\omega^2}\alpha\\ \alpha &= -\frac{\omega^2}{2\pi}\frac{\mathrm{d}T}{\mathrm{d}t}\end{aligned}$$

From a previous example $\omega = 7.29 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$

$$\alpha = -\frac{(7.29 \times 10^{-5})^2}{2\pi} 7.61 \times 10^{-13} = -6.44 \times 10^{-22}\,\mathrm{rad\,s^{-2}}$$